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The flow at a rear stagnation point is eventually determined by exponentially small values of the velocity

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It is suggested that current conceptions about unsteady rear-stagnation-point flow do not fully describe the physics, since they show discrepancies from recent numerical results. The previously neglected exponentially small rotational perturbation velocity above the boundary-layer proves to have a dominating influence on the final boundary-layer development. An asymptotic analysis reveals possible difficulties for common computational schemes for viscous flows. Failure of the usual asymptotic matching rule in the analysis is in accordance with Fraenkel's warning on logarithmic expansions.

1. Introduction

Unsteady flow at a rear stagnation point is one of the most basic problems in fluid mechanics (Schlichting 1979). In the framework of the Navier–Stokes equations, it is one of the relatively few exact solutions which retains the nonlinearity of these equations. In boundary-layer theory, the flow occurs whenever a smooth finite body is started from rest.

It is therefore alarming when a careful computation of the flow about an impulsively started rear stagnation point by Williams (1982) shows serious discrepancies from the semi-analytical results of Proudman & Johnson (1962), and its extension to higher order of approximation by Robins & Howarth (1972).

In a different context, in connection with unsteady boundary-layer separation, we became interested in this problem and decided to try to clarify the differences by a Lagrangian computation. The advantages of Lagrangian coordinates at a rear stagnation point had already been pointed out by Van Dommelen & Shen (1980). Additional accuracy was possible by restricting the attention solely to the rear stagnation point. The Lagrangian scheme is identical with the one in Van Dommelen (1983) for solving the flow at the meridional plane of the impulsively spun sphere. The computational aspects are therefore omitted in the following. Only the conclusion is of importance: our Lagrangian computation fully supports the deviations reported by Williams.

Table 1 compares the Lagrangian results at various mesh sizes, in order to convince ourselves that the deviations are not due to numerical errors. The results of Robins & Howarth (1972) agree with those of Howarth (Hommel 1983) using a full Navier–Stokes scheme. Also shown in table 1 are results obtained by Hommel (1983)

t	129	257	513	129/257	257/513	129/257/513	Hommel 1983	Howarth 1981	Cowley 1983
0.5	−0.188547	−0.188451	−0.188427	−0.188419	−0.188419	−0.188419	−0.188418	−0.18821	−0.188418
1.0	0.339921	0.340056	0.340090	0.340101	0.340101	0.340101	0.340102	0.34009	0.340102
1.5	0.666438	0.666423	0.666419	0.666418	0.666418	0.666418	0.666419	0.66647	0.666418
2.0	0.884792	0.884361	0.884252	0.884217	0.884216	0.884216	0.884219	0.88437	0.884216
2.5	1.023949	1.022778	1.022482	1.022388	1.022383	1.022383	1.02239	1.0226	1.022383
3.0	1.107823	1.105546	1.104969	1.104787	1.104777	1.104706	1.10479	1.1051	1.104775
3.5	1.157476	1.153700	1.152742	1.152441	1.152423	1.152421	1.1524	1.1528	1.15242
4.0	1.187998	1.182313	1.180869	1.180418	1.180388	1.180386	1.180	1.1808	1.1804
4.5	1.208404	1.200375	1.198334	1.197699	1.197654	1.197651	1.197	1.1981	1.1976
5.0	1.223461	1.212621	1.209865	1.209008	1.208946	1.208942	1.210	1.2094	1.209
5.5	1.235574	1.221421	1.217823	1.216703	1.216624	1.216618	1.220	1.2170	
6.0	1.246012	1.228009	1.223437	1.222008	1.221913	1.221907	1.22		
6.5	1.255526	1.233113	1.227434	1.225642	1.225541	1.225534	1.23		
7.0	1.264623	1.237223	1.230303	1.228090	1.227996	1.227990	1.23		
0.5	1.195730	1.195921	1.195969	1.195985	1.195985	1.195985	1.195974	1.1961	1.195985
1.0	2.601179	1.602284	2.602560	2.602652	2.602652	2.602652	2.6026	2.6067	2.602652
1.5	4.714062	4.718918	4.720131	4.720537	4.720535	4.720535	4.719	4.7385	4.720535
2.0	7.511341	7.529815	7.534421	7.535973	7.535956	7.535955	7.5	7.5938	7.535956
2.5	11.00184	11.06326	11.07852	11.08373	11.08361	11.08360	11.0	11.246	11.083600
3.0	15.56627	15.74827	15.79319	15.80893	15.80816	15.80811	15.0	16.231	15.8081
3.5	22.0089	22.5025	22.6229	22.6670	22.6631	22.6629	20.0	23.708	22.662
4.0	31.500	32.756	33.057	33.174	33.157	33.156		35.614	33.15
4.5	45.569	48.627	49.341	49.647	49.579	49.574		55.129	49.5
5.0	66.12	73.31	74.93	75.70	75.47	75.46		87.558	75
5.5	95.11	111.64	115.24	117.15	116.44	116.39		141.860	
6.0	133.6	171.0	178.8	183.5	181.4	181.3			
6.5	178.6	262.6	279.2	290.6	284.7	284.3			
7.0	211.0	403.0	438.0	467.0	449.0	448.0			

TABLE 1. Results for the wall shear, the displacement thickness δ^* according to computations with 129, 257 and 513 mesh points across the Lagrangian boundary layer. The time step ranged from 0.025 to 0.00625. Columns 5–7 describe the Richardson extrapolates of the indicated meshes. The last three columns describe values from the literature (Hommel 1983, Cowley 1983).

and by Cowley (1983), both of whom arrived at their results by extending the Blasius small-time expansion to high order of accuracy.

For moderate times, all results are in excellent agreement, but when the boundary layer becomes thick differences start to show. Our results support those of Cowley and Williams, who provided data up to $t = 5$. In particular we agree with Williams that the numerical results deviate from the semi-analytical curve given by Robins & Howarth.

The question arises whether the discrepancy can be eliminated by mere adjustment of the undetermined constants in the analytical curve, or that modification of the theory itself is required. In this paper, we will argue that conceptual modifications in the theory are in fact necessary.

2. The Proudman & Johnson theory

The stagnation point flow will be described in an (X, Y)-axis system with the Y-axis aligned with the axis of symmetry; the potential-flow velocity gradient is used to non-dimensionalize the time coordinate. The stream function may be written as $\Psi = -X\,Re^{-\frac{1}{2}}\psi(y,t)$ and the velocity parallel to the wall as $U = -Xu(y,t)$, where $y = Re^{\frac{1}{2}}\,Y$ and Re is the Reynolds number. In this notation, the Navier–Stokes equations reduce to:

$$u_t - u^2 + \psi u_y = -1 + u_{yy}, \quad u = \psi_y, \tag{1a}$$

$$\psi(y,0) = y, \quad u(y,0) = 1 \quad (y \neq 0), \tag{1b}$$

$$\left.\begin{aligned} \psi(0,t) = 0, \quad u(0,t) = 0, \\ \psi(\infty,t) \sim y - \delta^*, \quad u(\infty,t) = 1. \end{aligned}\right\} \tag{1c}$$

If this is regarded as an unsteady boundary-layer problem, the 'displacement thickness'

$$\delta^* \equiv \int_0^\infty 1 - u\,\mathrm{d}y \tag{1d}$$

does not represent the true displacement of the fluid at the outer edge of the boundary layer (Moore & Ostrach 1957). Above the boundary layer, the particle paths follow from integration of the continuity equation as:

$$\left.\begin{aligned} y = C\,\mathrm{e}^t - y_0^*; \quad C = \text{constant}; \\ y_0^* \equiv \left[\int_0^t \delta^*\,\mathrm{e}^{-t'}\,\mathrm{d}t'\right]\mathrm{e}^t. \end{aligned}\right\} \tag{2a}$$

It is the integral y_0^* that corresponds to the true downward displacement of the fluid.

On account of these particle paths, it is convenient to introduce a reduced coordinate

$$\sigma \equiv y\,\mathrm{e}^{-t}, \tag{2b}$$

and a reduced displacement thickness

$$D \equiv \delta^*\,\mathrm{e}^{-t}. \tag{2c}$$

Thus the particle paths above the boundary layer become

$$\left.\begin{aligned} \sigma + \sigma_0^* \equiv \Delta\sigma = \text{constant}, \\ \sigma_0^* \equiv \int_0^t D\,\mathrm{d}t'. \end{aligned}\right\} \tag{2d}$$

The stagnation point flow (1 *a–c*) was first determined for small times by Blasius (1908). According to his solution, the flow near the wall would reverse direction at time $t \approx 0.7$. However, the reversed flow occurs only near the wall and will not affect the analysis of this paper, which is mainly concerned with the boundary-layer flow at large distances from the wall.

Possible self-consistent behaviour of the flow for large times was examined by Proudman & Johnson (1962), following the earlier linearized work of Moore (1958). According to Proudman & Johnson, the boundary-layer thickness would not tend to a finite limit for large times, but continue to grow exponentially. Further, there is no downward convection at the outer edge of the boundary layer to limit the growth of the boundary layer by viscous diffusion.

Yet when the boundary layer becomes thicker, the viscous effects do become correspondingly weaker. Proudman & Johnson proposed that the final stages would be inviscid, and that a definite shape of the velocity profile would emerge:

$$\left.\begin{aligned} &u \sim 1 - F_0'(s), \\ &s \equiv \frac{2y}{\delta^*} = \frac{2\sigma}{D} \quad (D = D(t)), \end{aligned}\right\} \tag{3}$$

where the shape function F_0' was still unknown and s is a suitably defined similarity variable. We have slightly modified the analysis of Proudman & Johnson to simplify later arguments; for a more precise discussion the reader is referred to the original reference.

The equations of motion (1) may be rewritten in terms of the similarity coordinate s as:

$$\left.\begin{aligned} &\psi \equiv y - \tfrac{1}{2}\delta^* F(s,t); \\ &FF'' + F'(2-F') = -D^{-1}D_t sF'' + F_t' - 4\,\mathrm{e}^{-2t}D^{-2}F'''; \\ &F \sim F_0 + \dots; \end{aligned}\right\} \tag{4}$$

where accents denote differentiation with respect to the similarity coordinate.

In solving for the velocity profile, Proudman & Johnson found that $D^{-1}D_t$ must tend to zero for large times. If this condition is not met, the velocity profile decays too slowly to allow a match with an external flow in which the vorticity is exponentially small. Only when $D^{-1}D_t$ is small does the solution decay exponentially:

$$\left.\begin{aligned} &F_0 = 2(1-\mathrm{e}^{-s}); \\ &F_0' = 2\,\mathrm{e}^{-s} \sim 1 - u \quad (D^{-1}D_t \sim 0) \end{aligned}\right\} \tag{5}$$

Proudman & Johnson now take this constraint to mean that the reduced displacement thickness D tends to a constant for large time. In a footnote, they add that the analysis is only approximate, and that there is no assurance that D will indeed approach a constant value. However, they do not elaborate and proceed to define a 'constant c' in their figure 2 which requires constant D. Neither do Robins & Howarth pursue the matter any further.

In figure 1, the computed values for $1/D^2$ are compared to the large-time values s_1^* and s_1 proposed by Proudman & Johnson and Robins & Howarth, respectively. The computed results do not appear to converge to either one of these values. And the higher-order approximations s_2 and s_3 found by Robins & Howarth do very little to improve the agreement for later times. The shown curve s_3 corresponds to the curve plotted by Robins & Howarth: in their analytical expression for this curve there

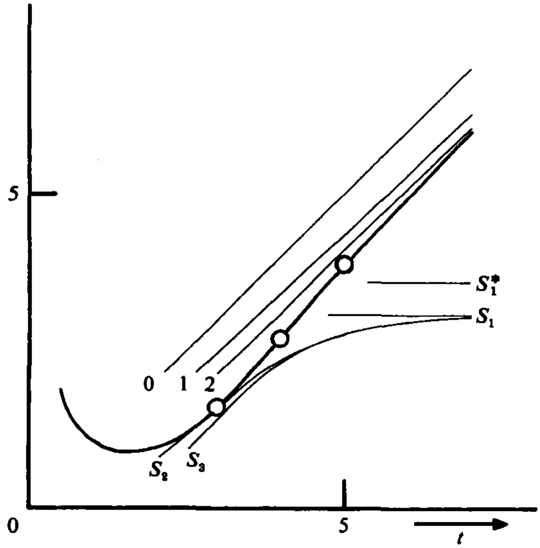

FIGURE 1. The quantity $1/D^2$, with D the reduced displacement thickness, (2c). Previous large-time theory predicts the curves s_1, s_2 and s_3, for increasing level of approximation; present theory predicts the curves 0, 1 and 2. The dots denote values according to Williams (1982).

appears to be a redundant factor 'c' which would increase the deviation from the computed results.

Yet, the condition of vanishing $D^{-1}D_t$ allows much more general asymptotic behaviour for D than just a constant. The result $D = O(t^{-\frac{1}{2}})$ suggested by figure 1 satisfies the condition, as does any algebraic blow-up or vanishing of D. Thus, there is a large indeterminacy in the expansion. This indeterminacy continues to higher order, although it has been eliminated by Robins & Howarth by the *ad hoc* assumption that the displacement effect of a viscous wall layer indicates the second term in the expansion.

However, we will show that the indeterminacy arises from a missing constraint in the Proudman & Johnson analysis: the solution should match the exponentially small rotational velocities above the boundary layer. In this additional matching, all indeterminacy is removed and we find $D \sim t^{-\frac{1}{2}}$ instead of being constant. Further the second term in the expansion is found to be algebraically small rather than exponentially small as proposed by Robins & Howarth.

3. Present theory

An aspect of the Proudman & Johnson solution which appears to have been overlooked previously is the surprisingly slow exponential decay of the velocity profile toward the external flow value $u = 1$. While their result has

$$1-u_0 = 2\exp\left(\frac{-2\sigma}{D}\right) \quad \left(s = \frac{2\sigma}{D} \gg 1\right), \tag{6a}$$

the Blasius small-time profile decays considerably more rapidly:

$$1-u \sim \exp(-|K|\sigma^2)[o(1)+\ldots] \quad (\sigma \to \infty), \tag{6b}$$

where $o(1)$ stands for a small algebraic factor which is not of importance at this stage and $K = K(t)$.

After this study had been completed, we learned of the linearized analysis of Moore (1958), which does show a stronger rate of decay similar to (6*b*). In fact, our proposed asymptotic expansions of the velocity profile, (11*a*, *b*) and (24) will agree with Moore's linearized solution. Thus the Proudman & Johnson nonlinear result becomes even more surprising. Inclusion of the exponentially small velocities above the boundary layer in the analysis is clearly indicated in order to settle the point.

Our approach will be first to derive a self-consistent expansion for the exponentially small velocities at finite times and at large distance σ from the wall. The undetermined constants in this expansion can be found from matching with the known Blasius small-time expansion. The expansion cannot directly be matched with the Proudman & Johnson solution for large times, but we will introduce a transition layer in which the velocity is described as an exponentially small perturbation of the potential flow. At its lower side the transition layer matches the Proudman & Johnson velocity profile and at its upper side the large σ-expansion. It describes, therefore, the transition from the slower decay (6*a*) to the faster decay (6*b*).

Now, the usual method of matched asymptotic expansions is not very suitable for handling exponentially small terms. It is convenient, therefore, to take the logarithm and define the reduced velocity v as:

$$v \equiv \ln(1-u) \quad (v = v(\sigma, t)). \tag{7}$$

Apparently, such a simple redefinition of the dependent variable is not sufficient for the class of problems addressed by Lange (1983), but must there be supplemented by conservation of energy.

The Proudman & Johnson similarity solution predicts a linear asymptotic behaviour of the reduced velocity:

$$v \sim -\frac{2\sigma}{D} \quad \left(s = \frac{2\sigma}{D} \rightarrow \infty\right), \tag{8}$$

valid when s, but possibly not σ, is large. Our proposal is a quadratic dependence for truly large σ:

$$v \sim K\sigma^2, \quad K = K(t) < 0 \quad (\sigma \rightarrow \infty). \tag{9}$$

The motivation for our proposal becomes evident when the governing equations of motion (1) are rewritten in terms of the reduced variables σ and v:

$$\left.\begin{aligned} v_t - \Phi v_\sigma - 2 + \mathrm{e}^v &= \mathrm{e}^{-2t}(v_\sigma^2 + v_{\sigma\sigma}); \\ \Phi &= \int_0^\sigma \mathrm{e}^v \, \mathrm{d}\sigma' \sim D \quad (\sigma \rightarrow \infty); \end{aligned}\right\} \tag{10a}$$

$$\Phi(\sigma, 0) = 0, \quad v(\sigma, 0) = -\infty \quad (\sigma \neq 0); \tag{10b}$$

$$v(0, t) = 0, \quad v(\infty, t) = -\infty. \tag{10c}$$

The viscous terms in (10*a*) are the ones on the right-hand side multiplied by e^{-2t} and are here nonlinear. It is this nonlinearity which allows friction to alter the magnitude of v far from the wall from its original value $v = -\infty$ at $t = 0$.

Balancing of the nonlinear dissipation term v_σ^2 with the time–rate-of-change term v_t in (10*a*) is achieved by requiring

$$v \sim K_0 \sigma^2, \quad K_0 < 0 \quad (\sigma \rightarrow \infty),$$
$$K_{0t} = \mathrm{e}^{-2t} \, 4K_0^2. \tag{11a}$$

The ordinary differential equation (11*a*) may be integrated in closed form. The integration constant may be determined, since clearly the time that K_0 becomes

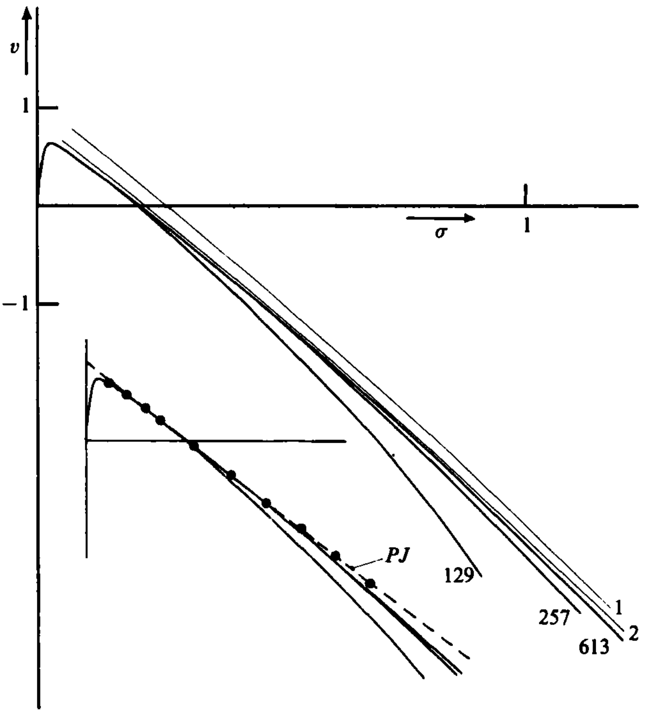

FIGURE 2. The reduced velocity v against the reduced coordinate σ, according to various results.

singular should be identified physically with the time of the impulsive start. Thus the solution is found to be

$$K_0 = \frac{-\frac{1}{2}}{1-e^{-2t}}. \tag{11b}$$

It may be verified that for small time these results describe the asymptotic behaviour of Blasius's error function profile:

$$v \sim \ln\left(\operatorname{erfc}\frac{\sigma}{2t^{\frac{1}{2}}}\right) \sim -\frac{\sigma^2}{4t}.$$

In summary, our proposal, (9), is that the reduced velocity should blow up quadratically rather than linearly for large σ. The coefficient of proportionality is determined by (11*b*). To verify this asymptotic behaviour numerically, however, is not as simple as it may seem, since the corresponding velocities are exponentially small. Our numerical results, shown in figure 2 for $t = 4.5$, include several curves to indicate the convergence with respect to mesh-size refinement.

Proudman & Johnson present data up to $\sigma = 0.6$, which follow the straight line PJ in figure 2. In order to avoid confusion with other curves, we have shifted the line PJ downward. Our results show excellent agreement with these data for $\sigma < 0.5$, but for larger values of σ our data continue to curve away from their straight line.

The numerical results in figure 2 cannot be represented by a straight line in the range shown. On the other hand, the proposed leading-order term (11*a*, *b*) presents an even poorer approximation to the numerical data, and there is no undetermined constant to force a better agreement. But, in the next section, the present theory will allow us to find higher-order approximations, (1) with an error of order $O(1/\Delta\sigma^2)$ and (2) with an error $O(1/\Delta\sigma^4)$. As shown in figure 2, these approximations do lead to excellent agreement with the numerical data.

Since curves 1 and 2 in figure 2 do not involve undetermined constants, we can feel confident that the exponentially small velocities above the boundary layer are correctly described by our theory, rather than by the Proudman & Johnson linear relationship. From an analytical point of view, a linear relationship seems further inconsistent with the Blasius small-time solution, since substitution into (10*a*) would suggest that the linear relationship would be valid for all times.

The quadratic relationship (11) also seems inconsistent with the Proudman & Johnson result, but we will show that a transition layer of exponentially small rotational velocities is present above the profile in which the quadratic asymptotic behaviour is achieved. We will start with deriving the equations which govern the motion in the transition layer, rather than to try to derive its structure immediately.

Above the boundary-layer region, the equations of motion (10) simplify since the e^v term is exponentially small. In addition, the boundary-layer-displacement effect may be eliminated completely. This simplification is achieved by definition of the coordinate

$$\Delta\sigma \equiv \sigma + \sigma_0^*. \tag{12a}$$

The simplified equation of motion,

$$v_t - 2 = e^{-2t}(v_{\Delta\sigma}^2 + v_{\Delta\sigma\,\Delta\sigma}), \tag{12b}$$

becomes linear when rewritten in terms of the true velocity:

$$u_t - 2(u-1) = e^{-2t}\, u_{\Delta\sigma\,\Delta\sigma}.$$

For large time, the viscous terms at the right-hand side of (12*b*) will be transcendentally small, provided that the scalings of v and $\Delta\sigma$ are algebraic in t. According to the results of this paper, this is in fact the case (the proper scalings are $v = t\tilde{v}$ and $\Delta\sigma = t^{\frac{1}{2}}\Delta\tilde{\sigma}$). It then follows from (12*b*) that to terms which are algebraically small, the transition layer is described by

$$v = 2t + V(\Delta\sigma) + \exp \quad (-v \gg 1). \tag{13}$$

Actually σ and $\Delta\sigma$ turn out to be $O(t^{\frac{1}{2}})$ and $v = O(t)$, so that the viscous terms in (12*b*) are indeed exponentially small.

According to the already known expansion for large σ, (11*a*, *b*), the still unknown function $V(\Delta\sigma)$ should expand for large values of its argument as:

$$V(\Delta\sigma) \sim -\tfrac{1}{2}\Delta\sigma^2. \tag{14}$$

The next step is to show that the above expansion for large $\Delta\sigma$ is also the required expansion of the transition layer for large time. To make the step, an *a priori* estimate is needed to the effect that the boundary-layer-displacement parameter σ_0^*, defined by (2*d*), becomes large. For, that will imply the required result,

$$\Delta\sigma > \sigma_0^* \gg 1.$$

In the analysis of Proudman & Johnson, the condition of large displacement is certainly satisfied, since σ_0^* would be $O(t)$. And while our numerical results in figure 1 indicate a more modest displacement thickness, they still integrate to $\sigma_0^* = O(t^{\frac{1}{2}})$.

The exponentially small velocities at large times are now found as the large $\Delta\sigma$-expansions (13), (14):

$$v \sim 2t - \tfrac{1}{2}\sigma_0^{*2} - \sigma_0^*\sigma - \tfrac{1}{2}\sigma^2 \quad (-v \gg 1,\ t \to \infty). \tag{15}$$

Because of the assumptions made, this result will become invalid in the similar region, where the velocity becomes finite. Thus, in the similar region the positive $2t$ term in (15) becomes comparable to the other terms:

$$2t \sim \tfrac{1}{2}\sigma_0^{*2}. \tag{16a}$$

The characteristic lengthscale will there be $\sigma = O(1/\sigma_0^*)$; in fact the similarity solution (8) is consistent with (15) if:

$$D \sim 2/\sigma_0^*. \tag{16b}$$

The similar region can only be consistent with the known exponentially small velocities above it when its displacement effect satisfies the constraints ($16a, b$). They determine the displacement effect as:

$$\left.\begin{aligned} \sigma_0^* &\sim 2t^{\frac{1}{2}} + \dots, \\ D = (\sigma_0^*)_t &\sim t^{-\frac{1}{2}} + \dots. \end{aligned}\right\} \tag{17}$$

The result for the reduced displacement thickness is shown as curve 0 in figure 1. Higher-order approximations are shown as 1, with an error $O(t^{-\frac{5}{2}} \ln^2 t)$ in D, and 2, with an error $O(t^{-\frac{7}{2}} \ln^3 t)$. Remarkably, all these approximations are free of undetermined constants. Not only the qualitative behaviour of the displacement thickness, but also the actual values may be determined analytically for large times!

The values of the wall shear may also be found analytically. The similar velocity profile (5) predicts a velocity near the wall which is precisely the opposite of the external flow velocity. Thus the inviscid flow near the wall resembles a front stagnation point rather than a rear one. As a consequence, near the wall a steady front-stagnation-point boundary layer develops (Proudman & Johnson 1962):

$$\psi \sim f_0(y) + \dots, \tag{18}$$

where f_0' describes the familiar Hiemenz flow profile, with a wall shear

$$f_0'' = -1.2326.$$

This prediction is in very good agreement with the numerical data, as shown in figure 3, but actually the prediction would be valid for virtually any reasonable assumption for the behaviour of the displacement thickness D.

The curves s_2^-, s_2^+ and s_3 represent increasingly accurate approximations to the wall shear according to Robins & Howarth. Yet their close agreement with the numerical results does not form as strong a vindication of the Proudman & Johnson analysis as it might seem. For, these higher-order approximations involve undetermined constants, and Robins & Howarth have chosen two of these constants to get the best agreement with the computed wall shear for curves s_2^+ and s_3.

The only undetermined constant in the curve s_2^- was found in another way, and this curve is in much poorer agreement with the numerical data. The curve s_2^- is also poorer than the present higher-order approximations 1 and 2, yet s_2^- should have an error of only $O(\mathrm{e}^{-\frac{3}{2}t})$, while the curve 1 has an error $O(t^{-\frac{1}{2}}\,\mathrm{e}^{-t})$ and the curve 2 an error of about $O(t^{-\frac{3}{2}}\,\mathrm{e}^{-t})$. Curves 1 and 2 do not involve undetermined constants, except that the usual perturbation problems to Hiemenz flow must be solved numerically to find them.

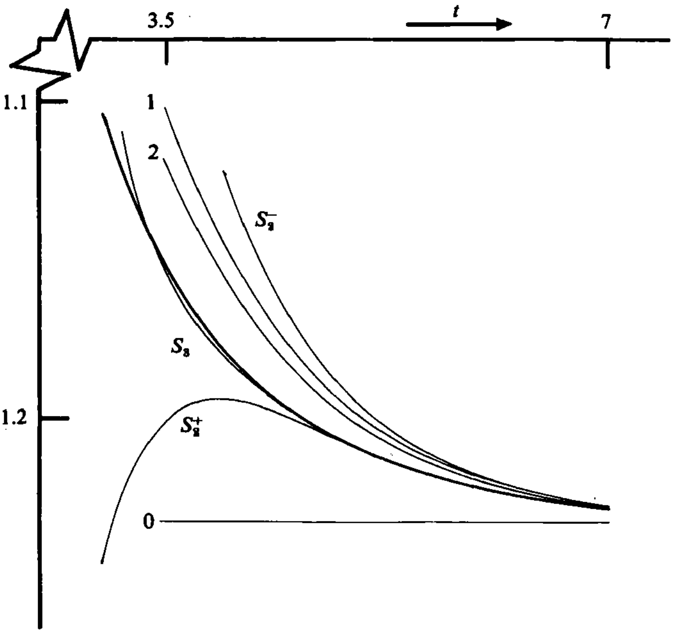

FIGURE 3. The wall shear according to previous and present theories.

4. More terms

The results will now be extended to higher orders of approximation. For small time, the solution to the reduced equations of motion (10) may be expanded as:

$$v \sim v_0(\eta) + tv_1(\eta) + t^2 v_2(\eta) + \ldots \quad \left(\eta = \frac{y}{2t^{\frac{1}{2}}}\right). \tag{19a}$$

According to Blasius (1908)

$$v_0 \sim \ln(\operatorname{erfc} \eta). \tag{19b}$$

It is convenient to rewrite v_1 and v_2 in terms of new functions ζ'_{01} and ζ'_{02}:

$$v_1 \equiv \frac{\zeta'_{01}}{\operatorname{erfc}}; \quad v_2 \equiv \frac{-\zeta'_{02}}{\operatorname{erfc}} - \tfrac{1}{2}v_1^2; \tag{19c}$$

since the solutions for the problems for ζ'_{01} and ζ'_{02} have already been given in the literature. In particular the solution for ζ'_{01} was due to Blasius. The solution for ζ'_{02} is very laborious and the results given by Goldstein & Rosenhead (1936) contain mistakes. In fact, these results would introduce $O(\sigma^4)$ and $O(\sigma^3)$ terms in the asymptotic expansion, invalidating all our analysis. But the results of Wundt (1955) are probably correct, in his notation. At least, the function ζ'_{02} tabulated there is correct to six digits. This was verified by a numerical solution of the ordinary differential equations satisfied by v_1 and v_2 which we carried out. Moreover, no inconsistencies were observed in the matching with the large σ-expansion.

Using Wundt's results, the small-time solution expands at the outer edge of the boundary layer as:

$$\left.\begin{aligned} v_0 &\sim -\eta^2 - \ln(\eta) + \ln(d_0) - \tfrac{1}{2}\eta^{-2} + \tfrac{5}{8}\eta^{-4} - \tfrac{37}{24}\eta^{-6} + O(\eta^{-8}); \\ v_1 &\sim \eta^2 - \tfrac{4}{3}d_0\,\eta + \tfrac{5}{2} - \tfrac{2}{3}d_0\,\eta^{-1} - 2(1+\tfrac{1}{3}d_0^2)\,\eta^{-2} + \tfrac{2}{3}d_0\,\eta^{-3} + 5(1+\tfrac{1}{3}d_0^2)\,\eta^{-4} + O(\eta^{-5}); \\ v_2 &\sim -\tfrac{1}{3}\eta^2 + \tfrac{4}{5}(d_0-d_1)\,\eta + \tfrac{1}{3}(\tfrac{1}{4}-\tfrac{4}{3}d_0^2) - \tfrac{2}{5}(\tfrac{2}{3}d_0+d_1)\,\eta^{-1} - \tfrac{1}{3}(1-\tfrac{2}{3}d_0^2)\,\eta^{-2} + O(\eta^{-3}); \\ & d_0 = \pi^{-\frac{1}{2}}, \quad d_1 = -d_0\left(1 - \frac{2^{\frac{5}{2}}}{3} + \frac{4}{9\pi}\right). \end{aligned}\right\} \tag{20}$$

The constants d_0 and d_1 correspond to the coefficients in the small-time expansion for the displacement thickness:

$$\delta^* \sim 2t^{\frac{1}{2}}[d_0 + td_1 + \ldots]. \tag{21}$$

Rewritten in terms of σ, the above results show that at the outer edge the boundary-layer flow expands as:

$$v \sim K_0\,\sigma^2 + K_1\,\sigma + K_2\ln(\sigma) + k_2 + \frac{K_3}{\sigma} + \frac{K_4}{\sigma^2} + \frac{K_5}{\sigma^3} + O\left(\frac{1}{\sigma^4}\right); \tag{22}$$

$$\left.\begin{aligned} K_0 &\sim -\tfrac{1}{4}t^{-1} - \tfrac{1}{4} - \tfrac{1}{12}t + \ldots; \\ K_1 &\sim -\tfrac{2}{3}d_0\,t^{\frac{1}{2}} - \tfrac{2}{5}(d_1+\tfrac{2}{3}d_0)\,t^{\frac{3}{2}} + \ldots; \\ K_2 &\sim -1 + o(t^2); \\ k_2 &\sim \tfrac{1}{2}\ln(4t) + \ln(d_0) + \tfrac{3}{2}t + \tfrac{1}{3}(\tfrac{1}{4}-\tfrac{4}{3}d_0^2)\,t^2 + \ldots; \\ K_3 &\sim -\tfrac{4}{3}d_0\,t^{\frac{3}{2}} + \tfrac{4}{5}(d_0-d_1)\,t^{\frac{5}{2}} + \ldots; \\ K_4 &\sim -2t - 4(1+\tfrac{2}{3}d_0^2)\,t^2 + 8(\tfrac{4}{3}+\tfrac{7}{9}d_0^2)\,t^3 + \ldots; \\ K_5 &\sim \tfrac{16}{3}d_0\,t^{\frac{5}{2}} + \ldots. \end{aligned}\right\} \tag{23}$$

Additional simplifications are possible by rewriting the expansion in terms of $\Delta\sigma$:

$$v \sim L_0\,\Delta\sigma^2 + L_1\,\Delta\sigma + L_2\ln(\Delta\sigma) + l_2 + \frac{L_3}{\Delta\sigma} + \frac{L_4}{\Delta\sigma^2} + \frac{L_5}{\Delta\sigma^3} + O\left(\frac{1}{\Delta\sigma^4}\right). \tag{24}$$

Substitution in the equation of motion (12*b*) leads to a system of ordinary differential equations for the L_t and l_2. The solution matches the small-time expansion (23) for suitable values of the integration constants. Thus it is found that $L_1 = L_3 = L_5 = 0$ and

$$\left.\begin{aligned} L_0 &= -\frac{\frac{1}{2}}{1-\mathrm{e}^{-2t}}, \quad L_2 = -1, \\ l_2 &= -\tfrac{1}{2}\ln|L_0| + \ln(d_0) + 2t, \\ L_4 &= \tfrac{1}{2}L_0^{-1} - (\tfrac{3}{8}+\tfrac{1}{6}d_0^2)\,L_0^{-2}. \end{aligned}\right\} \tag{25}$$

The expansion (24) with an error $O(\ln\Delta\sigma)$ falls outside the boundaries of figure 2. The expansion to error $O(1/\Delta\sigma^2)$ is shown as curve 1 and to error $O(1/\Delta\sigma^4)$ as curve 2.

For large time, the above results show that the asymptotic expansion for v proceeds as:

$$\begin{aligned} v &= 2t + V(\Delta\sigma) + \exp, \\ &\sim -\tfrac{1}{2}\Delta\sigma^2 - \ln(\Delta\sigma) + \ln(2^{\frac{1}{2}}d_0) + 2t - \frac{(\frac{5}{2}+\frac{2}{3}d_0^2)}{\Delta\sigma^2} + O\left(\frac{1}{\Delta\sigma^4}\right). \end{aligned} \tag{26}$$

It is here possible to introduce formal new variables $\sigma/t^{\frac{1}{2}}$ and $\sigma_0^*/t^{\frac{1}{2}}$, but the bottom line is again that the above expansion for large $\Delta\sigma$ is equivalent to the expansion for large time of the flow in the transition layer.

The obtained solution (26) expands near the wall as:

$$v \sim 2t-\tfrac{1}{2}\sigma_0^{*2}-\sigma_0^*\sigma-\tfrac{1}{2}\sigma^2-\ln(\sigma_0^*)+\ln(2^{\frac{1}{2}}d_0)-\sigma_0^{*-1}\sigma+\ldots-(\tfrac{5}{2}+\tfrac{2}{3}d_0^2)\sigma_0^{*-2}+\ldots. \quad (27)$$

In order that this result be matchable with the similarity profile (6*a*), the similar region should generate a displacement parameter σ_0^* satisfying

$$2t-\tfrac{1}{2}\sigma_0^{*2}-\ln(\sigma_0^*)+\ln(2^{\frac{1}{2}}d_0) \sim \ln(2), \quad (28a)$$

while its characteristic dimension should be

$$\tfrac{1}{2}D \sim 1/\sigma_0^*. \quad (28b)$$

The displacement parameter follows from (28*a*) as

$$\sigma_0^* \sim 2t^{\frac{1}{2}}-\tfrac{1}{2}t^{-\frac{1}{2}}\{\ln(2t^{\frac{1}{2}})-\ln(d_0/2^{\frac{1}{2}})\}+\ldots. \quad (29a)$$

The displacement thickness is described as the derivative of σ_0^*, hence,

$$D \sim t^{-\frac{1}{2}}+\tfrac{1}{4}t^{-\frac{3}{2}}[\ln(2t^{\frac{1}{2}})-\ln(d_0/2^{\frac{1}{2}})-1], \quad (29b)$$

which verifies the requirement (28*b*). This result is shown as curve 1 in figure 1.

Before proceeding to higher order, the coordinate s will be redefined slightly:

$$s \equiv \sigma_0^*\sigma. \quad (30)$$

Furthermore, we will use σ_0^* as the independent variable instead of t, so that t will now be regarded as a function of σ_0^*. The main advantage is that the logarithmic term differentiates away in (28*a*), but not in (29*a*).

The transformed equations of motion (4) become

$$FF''+F'(2-F') = \frac{\sigma_0^* F'_{\sigma_0^*}+sF''}{\sigma_0^* t_{\sigma_0^*}}-e^{-2t}\sigma_0^{*2}F'''. \quad (31a)$$

There are two matching processes involved: the finite part of F should match the boundary-layer-displacement parameter σ_0^*, while the exponentially small terms should match the exponentially small velocity e^v. Rewritten, this yields

$$F \sim \sigma_0^* D = \sigma_0^*/t_{\sigma_0^*}, \quad \ln(F') \sim v. \quad (31b)$$

The first matching requires, on account of (28),

$$F \sim 2-2\sigma_0^{*-2}+\ldots, \quad (32a)$$

and the second, according to (27),

$$\ln(F') \sim 2t-\tfrac{1}{2}\sigma_0^{*2}-s-\tfrac{1}{2}\sigma_0^{*-2}s^2$$
$$-\ln(\sigma_0^*)+\ln(2^{\frac{1}{2}}d_0)-\sigma_0^{*-2}s-(\tfrac{5}{2}+\tfrac{2}{3}d_0^2)\sigma_0^{*-2}+O(\sigma_0^{*-4}). \quad (32b)$$

Since the homogeneous perturbation solutions to (31*a*) have non-zero values at large s, the first term in the further expansion of F must be:

$$F \sim F_0+\sigma_0^{*-2}F_1+\ldots. \quad (33a)$$

Substitution in (31*a*) shows

$$F_1 = 2[s\,e^{-s}+e^{-s}-1][1+\ln(1-e^{-s})]+\left[s^2+2s-2\int_0^s \frac{s'}{e^{s'}-1}\,ds'\right]e^{-s}. \quad (33b)$$

The reduced velocity follows as

$$\ln(F') = \ln(F'_0) + \sigma_0^{*-2}\frac{F'_1}{F'_0} + \ldots \sim -s + \ln(2) + \sigma_0^{*-2}[\tfrac{1}{6}\pi^2 - \tfrac{1}{2}s^2 - s]. \tag{34}$$

This matches (32*a*) if

$$2t - \tfrac{1}{2}\sigma_0^{*2} - \ln(\sigma_0^*) + \ln(2^{\frac{1}{2}}d_0) - (\tfrac{5}{2} + \tfrac{2}{3}d_0^2)\,\sigma_0^{*-2} \sim \ln(2) + \tfrac{1}{6}\pi^2\sigma_0^{*-2}. \tag{35a}$$

The third-order expression for the displacement thickness may now be found from inversion and differentiation:

$$\left.\begin{aligned} D &\sim t^{-\frac{1}{2}} + \tfrac{1}{4}t^{-\frac{3}{2}}D_1 + \tfrac{1}{32}t^{-\frac{5}{2}}[3D_1^2 - 2D_1 + \pi^2 + 4d_0^2 + 12] + \ldots;\\ D_1 &\equiv \ln(2t^{\frac{1}{2}}) - \ln\left(\frac{d_0}{2^{\frac{1}{2}}}\right) - 1. \end{aligned}\right\} \tag{35b}$$

This result is shown as curve 2 in figure 1.

Near the wall, the flow in the nonlinear region expands as

$$u = 1 - F' \sim -1 + 2s[1 + \sigma_0^{*-2}\ln(s) + \ldots] + s^2[\ldots], \tag{36}$$

since all higher-order perturbations to F' start with a factor s. In this case, the asymptotic matching rule will prove to fail. In order to avoid this difficulty, a more detailed description must be given of the inviscid-flow region at exponentially small s, but still above the Hiemenz layer.

From a physical point of view, there seems little reason to doubt the suggestion by (36) that the flow will be described as an exponentially small perturbation of an $u \equiv -1$ front-stagnation-point flow. In other words, we postulate the governing equations to be

$$\psi = -\Delta y + \exp, \quad \Delta y \equiv y - \delta_H^*, \quad u = -1 + u', \tag{37a}$$

where the exponentially small perturbation velocity u' satisfies the equations of motion:

$$u'_t = \Delta y\, u'_{\Delta y} - 2u'. \tag{37b}$$

Subtraction of the Hiemenz-flow displacement thickness δ_H^* is a sensible simplification, since the only change it requires in the problem is that the wall boundary condition is satisfied at $\Delta y = -\delta_H^*$ rather than at $y = 0$. It avoids having to carry along the value of δ_H^* at each subsequent step.

The solution to the perturbation problem (37) is of the general form:

$$u' = e^{-2t}f(p), \quad p \equiv \Delta y\, e^t = \frac{s\, e^{2t}}{\sigma_0^*}. \tag{38}$$

The requirement that this result be consistent with the obtained expansion (36) for small s determines the behaviour of the function f for large values of its argument:

$$f(p) \sim 2p\left\{[2\ln(p)]^{\frac{1}{2}} + [2\ln(p)]^{-\frac{1}{2}}\ln\left(\frac{d_0}{2^{\frac{1}{2}}}\right) + \ldots\right\}. \tag{39}$$

But, for a result described by such an expression, the asymptotic matching rule cannot be used. Indeed, if we introduce an infinity of 'intermediate' coordinates,

$$\tau \equiv e^{-\alpha t}\,\Delta y \equiv e^{-(\alpha+1)t}\,p \quad (0 < \alpha < 1),$$

then the results (38), (39) expands differently in each region, namely, rewritten in terms of Δy as

$$u' = 2\,\Delta y\, e^{-t}\{[2(1+\alpha)\,t]^{\frac{1}{2}} + \ldots\}.$$

Use of the asymptotic matching rule is equivalent to expanding the function $f(p)$ in the s-region, where $\alpha = 1$, then rewriting it in terms of Δy to the above result with a factor $[4t]^{\frac{1}{2}}$. But directly expanding $f(p)$ in the Δy-region, where $\alpha = 0$, will give the correct result $[2t]^{\frac{1}{2}}$. Note the insidiousness of the failure: the result is of the correct sign and only wrong by a factor of $2^{\frac{1}{2}}$. If it is compared with the computed wall shear, one would be tempted to suppose that the computed times are too low for precise agreement.

The reason for the failure is clear, of course: the expansion (36) is really an expansion in powers of logarithms in the notation of Fraenkel (1969). In that notation, the small variable is defined as the ratio of the inner and outer scales, i.e. as e^{-t}. The suggestion by Lo (1983) to avoid the difficulty seems less helpful here, since we can evaluate neither of the Fraenkel operators. Our approach in postulating the governing equations (37) follows the ideas of Kaplun (1967).

Turning the attention then, finally, to the Hiemenz layer, the results (38), (39) expand at algebraically large Δy as:

$$u \sim -1+2\,\Delta y\, e^{-t}\{[2t]^{\frac{1}{2}}+[2t]^{-\frac{1}{2}}[\ln(\Delta y)+\ln(d_0/2^{\frac{1}{2}})]+\ldots\}+\ldots. \qquad (40a)$$

Thus in the Hiemenz layer the flow expands further as:

$$\psi \sim f_0(\Delta y)+e^{-t}\{[2t]^{\frac{1}{2}}f_{10}(\Delta y)+[2t]^{-\frac{1}{2}}f_{11}(\Delta y)+\ldots\}+\ldots. \qquad (40b)$$

By Runge–Kutta solution of the perturbations f_{10} and f_{11}, the wall shear is found as

$$\psi_{yy} \sim -1.2326+e^{-t}\,1.6337\left\{[2t]^{\frac{1}{2}}+[2t]^{-\frac{1}{2}}\left[\ln\left(\frac{d_0}{2^{\frac{1}{2}}}\right)-0.0177\right]+\ldots\right\}+\ldots. \qquad (41)$$

This result is shown to increasing order of approximation as curves 0, 1 and 2 in figure 3. The curve 2 reduces the remaining discrepancy by only 29% at $t = 4.5$, but this has already improved to 39% at $t = 7$.

Notice the remarkable result that the coefficient d_0 in the small-time expansion reappears here in the large-time expansion for the wall shear!

5. Conclusion

A self-consistent, large-time description for the unsteady flow at a rear stagnation point was first proposed by Proudman & Johnson (1962) and generalized by Robins & Howarth (1972). In the expansions, a considerable amount of indeterminacy arose, which was eliminated by means of *ad hoc* assumptions. However, these assumptions do not lead to good agreement with computed results for the displacement thickness. As an example, figure 1 shows that the quantity e^{2t}/δ^{*2} does not tend to the expected constant value, at least not for $0 < t < 7$.

According to our analysis, the reason is that Proudman & Johnson matched their velocity profile at its upper edge only up to algebraically small terms with the 'potential flow' above. We find it necessary that the matching also includes the transcendentally small rotational velocities above the boundary layer.

By means of this additional matching, all indeterminacy in the expansions did in fact disappear. The reason was that the transcendentally small rotational velocities are governed by a linearized problem, (12c), and could be found analytically for all times.

It is a remarkable result that according to our analysis, the transcendentally small rotational velocities dominate the eventual mechanics of the boundary layer. This is possible on account of the expression (13) for the streamwise velocity gradient u above the boundary layer:

$$u = 1 - e^{2t} f\{(y + y_0^*) e^{-t}\} \quad (1 - u \ll 1) \tag{42}$$

which shows that the rotational part of u grows proportional to e^{2t}. Further, since $y_0^* \sim 2t^{\frac{1}{2}} e^t$ these growing disturbances are transported downward toward the Proudman & Johnson main boundary layer below.

This result, that the Navier–Stokes and boundary-layer equations may amplify originally very small velocities, must be a concern in numerical schemes. In boundary-layer computations, a common numerical approach enforces the external flow boundary condition at a finite distance from the wall. Clearly, that is equivalent to setting the exponentially small velocities at the outer edges of the boundary layer to zero. But then the computation would no longer contain the very information which determines the solution for later times.

Another difficulty in an Eulerian computation is the need to differentiate the function $f\{\ \}$ in (42) numerically. The asymptotic shape of this function is

$$\exp\{-\tfrac{1}{2} e^{-2t}(y + y_0^*)^2\},$$

and, to discretize such a function, the required resolution increases away from the wall. Indeed, the characteristic lengthscale is found as

$$O\left(\frac{e^{2t}}{y + y_0^*}\right) = O\left(\frac{1}{y}\right).$$

Our Lagrangian computation has the advantage that there is no need to evaluate the spatial derivative in the convective term.

It would seem plausible that the deviations between various computations at larger times (table 1, figure 2) could at least partly be due to the above difficulties.

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